Structure or Noise?

NSF Workshop on
Uncertainty in Complex Interacting Systems

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Why We Must Model I

- Nature spontaneously organizes

Emergent structures

Why We Must Model 2

- Engineered systems spontaneously organize
  - Internet route flapping
  - Power-law Internet organization
  - Financial markets crash
  - Power grids fail spectacularly
  - Social pattern formation on the web
  - ...

And so ...

- Problem:

  Emergent structures not given directly by the
  system coordinates or
  governing equations of motion

- Consequence:

  Each system needs its own explanatory basis

Why we must Model 3

Fundamental: Mathematics of Intrinsic Randomness

- Nonlinear dynamical systems [Kolmogorov 1958]:

  Chaotic systems: Shannon entropy $h_\mu > 0$

- Kolmogorov-Chaitin complexity of Data [1963]:

Size of shortest Turing Machine Program to predict Data

- KC complexity = f(Shannon entropy) [Brudno 1978] :

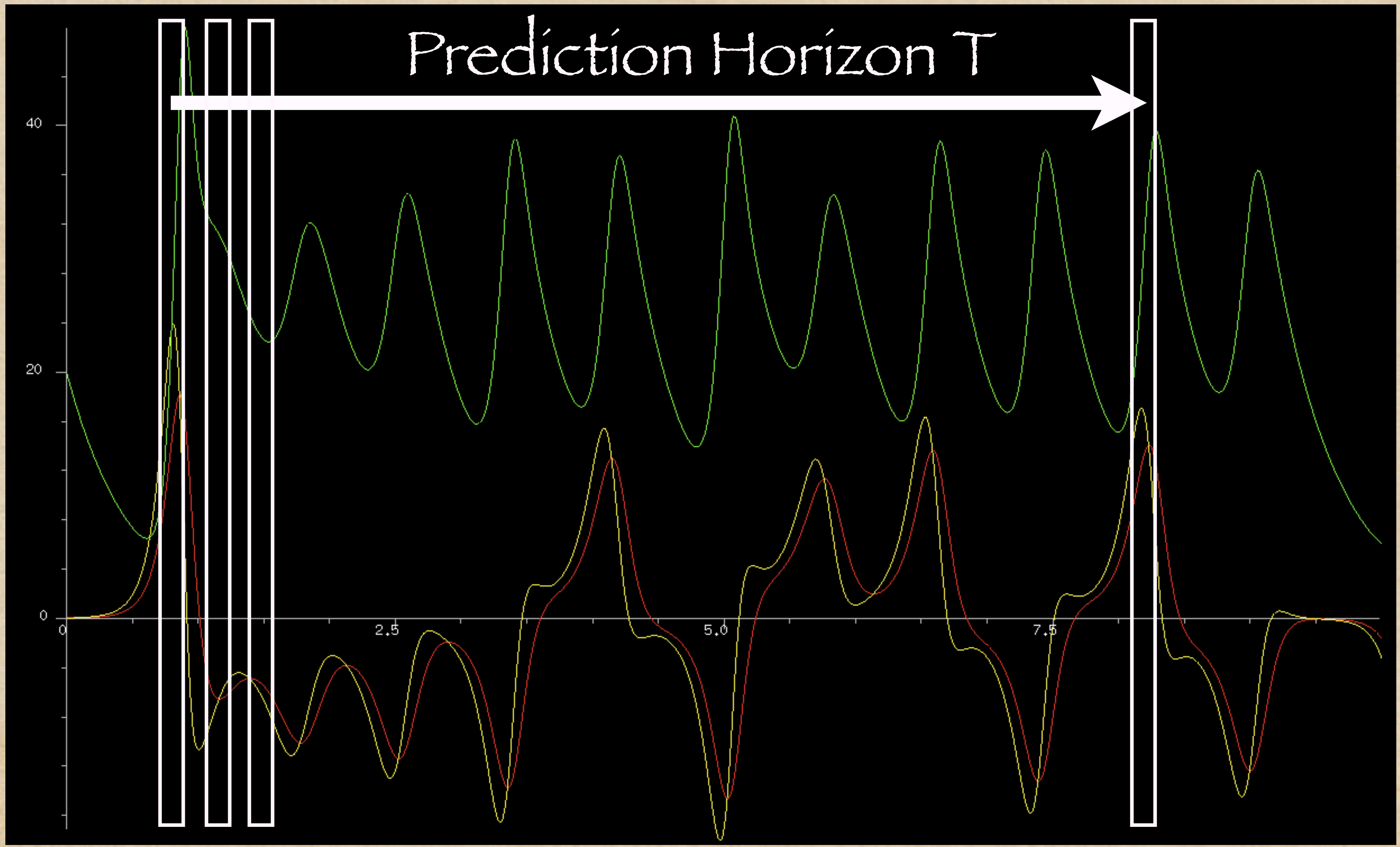
$$|\mathrm{Program}| \propto e^{h_\mu |\mathrm{Data}|}$$

Exponential Increase in Prediction Resources

$$\text{Accuracy} \propto e^{-T}$$

$$|\text{Measurements}| \propto e^{T}$$

$$|\text{Compute time}| \propto e^{T}$$



Consequence

- No short cuts!
  - No closed-form solutions
  - No computational speed-ups
  - Must compute full trajectory

- Right representation is critical for reducing the prediction error as far as possible (but no further!)
- Cannot estimate uncertainty unless you know the right model.

Fundamental in Complex Systems!

- Each nonlinear system requires its own representation
- Selecting balance between ascribing structure or noise to a measurement depends on representation

- Fundamental issue: Theory building
- Subsidiary issue:

  Statistical fluctuations due to finite data sample

  (This is NOT a talk about machine learning.)

Learning a Model

- Problem: Learn from Observations
  - The world behaves: $\overleftrightarrow{X} = \underset{\text{past}}{\overleftarrow{X}} \; \underset{\text{future}}{\overrightarrow{X}}$
    - Given $\Pr(\overleftarrow{X}, \overrightarrow{X})$, agent learns model:

      States $\mathcal{R}$ and Dynamic

- Pattern Discovery:

  Learn the world's hidden states $\Pr(\mathcal{R} | \overleftarrow{X})$

JPC & K. Young, Inferring Statistical Complexity, Physical Review Letters 63 (1989) 105-108.
C. R. Shalizi & JPC, Journal Statistical Physics 104 (2001) 817-879.

Learning a Model ...

- Causal shielding:

$$\Pr(\overleftarrow{X}\overrightarrow{X}) = \Pr(\overleftarrow{X}\,|\mathcal{R})\Pr(\overrightarrow{X}\,|\mathcal{R})$$

- Dynamics of learning:

  Search in the space of models: $\mathcal{R} \in \mathcal{M}$

Learning a Model ...

(Susanne Still, Chris Ellison, & JPC)

- Causal shielding objective function

$$\min_{\Pr(\mathcal{R}|\overleftarrow{X})} \left(I[\overleftarrow{X};\mathcal{R}] + \beta I[\overleftarrow{X};\overrightarrow{X}\,|\mathcal{R}] \right)$$

Model: Map from histories to states	Info states contain about histories	Reduce info history has about future

- Structure/Noise Design Control: $\beta \sim 1/T$

arxiv.org: 0708.0654 [physics.gen-ph] & 0708.1580[cs.IT]

Learning a Model ...

- Optimal states $\Pr(\mathcal{R}|\overleftarrow{X})$ are Gibbs states:

$$\Pr_{\text{opt}}(\mathcal{R}|\overleftarrow{X}) = \frac{\Pr(\mathcal{R})}{Z(\overleftarrow{X},\beta)} e^{-\beta E(\mathcal{R},\overleftarrow{X})}$$

where

$$E(\mathcal{R},\overleftarrow{X}) = \mathcal{D}\left(\Pr(\overrightarrow{X} \mid \overleftarrow{X}) || \Pr(\overrightarrow{X} | \mathcal{R})\right)$$

$$\Pr(\overrightarrow{X} | \mathcal{R}) = \frac{1}{\Pr(\mathcal{R})} \sum_{\overleftarrow{X}} \Pr(\overrightarrow{X} \mid \overleftarrow{X}) \Pr(\mathcal{R}|\overleftarrow{X}) \Pr(\overleftarrow{X})$$

$$\Pr(\mathcal{R}) = \sum_{\overleftarrow{X}} \Pr(\mathcal{R}|\overleftarrow{X}) \Pr(\overleftarrow{X})$$

Learning a Model ...

- Solve these equations self-consistently

  (Analytical in special cases; numerical generally)

- Parametrized family of models:

  $$R_\beta \colon \Pr(\mathcal{R} | \overleftarrow{X})$$

- Structure or Noise?

  β trades-off model size against prediction error

  Structure versus Noise

What Do Solutions Mean?

Causal Models

- Causal architecture given by ϵ-Machine M:
  - Optimal predictor:

$$h_\mu(M) \leq h_\mu(\mathcal{R})$$

  - Minimal size (within optimal predictors $\widehat{\mathcal{R}}$):

$$C_\mu(M) \leq C_\mu(\widehat{\mathcal{R}})$$

  - Unique (within min, opt predictors)

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C. R. Shalizi & JPC, Journal Statistical Physics 104 (2001) 817-879.

Learning a Model ...

- Theorem: Low-temperature limit

$$\beta \rightarrow \infty$$

Recover ϵ-Machine:

$$R_\beta \rightarrow M$$

- Conclusion:

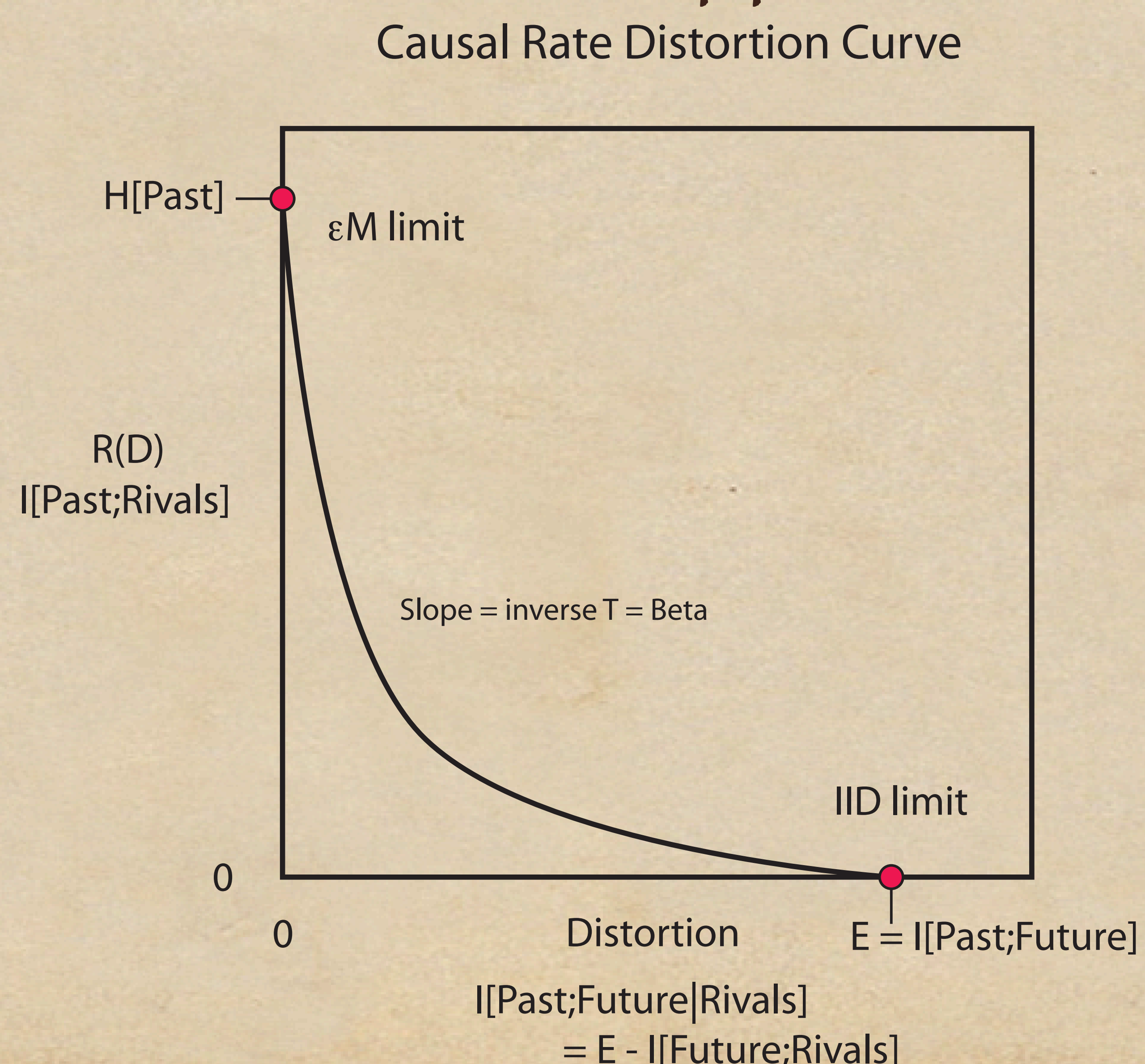
At given prediction error

R_β is best causal approximate.

Learning a Model ...

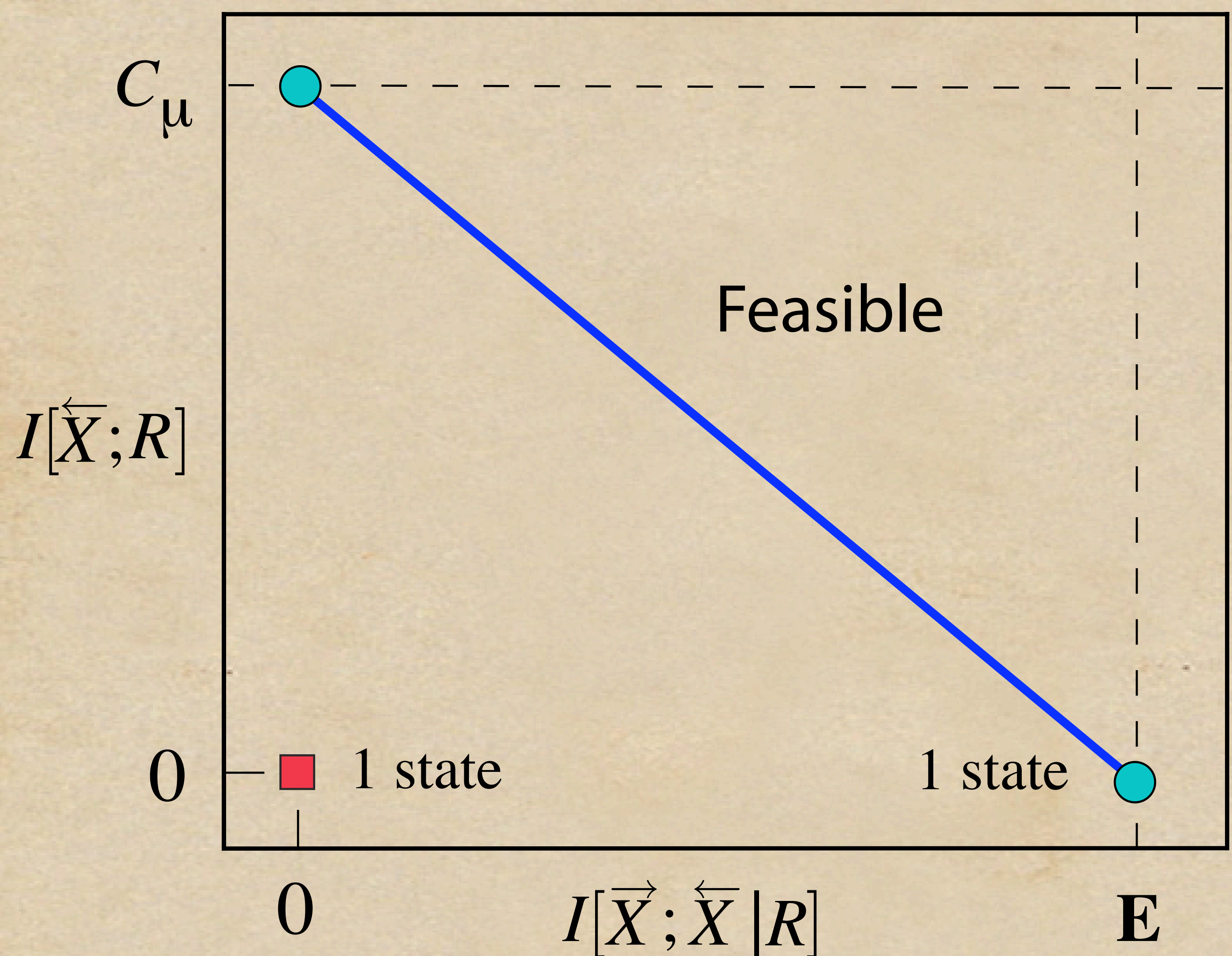
Optimally balance structure & error
At each level β of approximation

In theory



Learning a Model ...

Analytical cases



Predictively Reversible:

$$P(\overrightarrow{x} \mid \overleftarrow{x}) = \delta_{\overrightarrow{x}, f(\overleftarrow{x})}$$

(e.g., periodic)

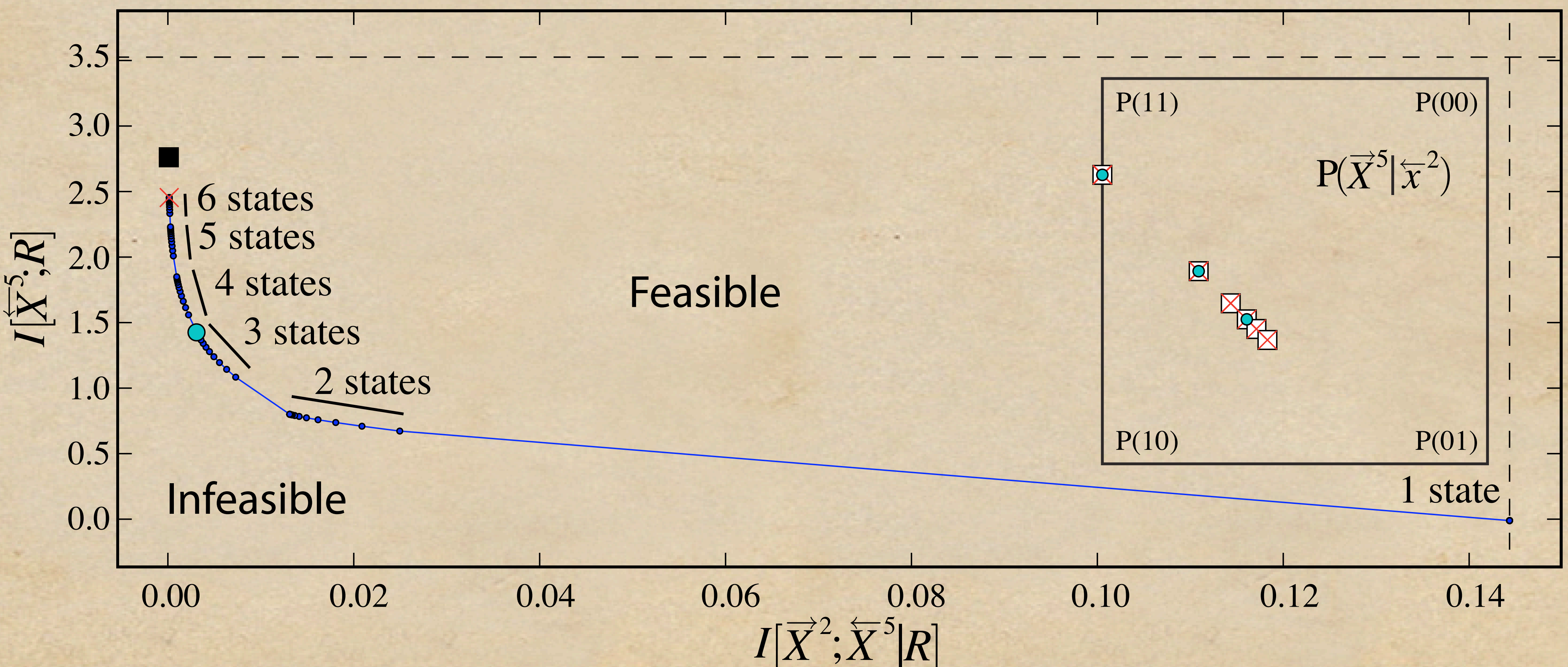
All IID processes:

$$P(\overrightarrow{x} \mid \overleftarrow{x}) = P(\overrightarrow{x})$$

Learning a Model ...

Optimally balance structure & error
At each level β of approximation

In practice: Learn an oo-state world (SNS: "simple nondeterminstic source")



Learning a Model ...

- Causal compressibility: Shape of RD curve
  - Benefit of choosing smaller model for loss in predictability
  - Deviation from straight-line RD curve
- IID: No
- Predictively reversible: No
- SNS: Yes

Conclusions

- Causal shielding principle leads to
  - Process's organization: ϵ-Machine
  - Family of best approximations to ϵ-M
- Structure or Noise?
  - Cannot estimate uncertainty until you know structure.
  - First-principle distinction
  - Principled trade-off

Thanks!

All papers online at the
Computational Mechanics Archive
http://cse.ucdavis.edu/~cmg/